

# Entropy is complexity

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## Abstract

In a recent paper Andrei N. Soklakov explained the foundations of the Lagrangian formulation of classical particle mechanics by means of Kolmogorov complexity. In the present paper we use some of Soklakov ideas in order to derive the second law of thermodynamics. Our main result is that the complexity of a thermal system corresponds to its entropy.

## 1 Introduction

By means of the so-called prefix version of Kolmogorov complexity, introduced by Levin [3, 4], Gács [2], and Chaitin [1], Soklakov [5] was able to explain why the Lagrangian  $L$  of a composite system always has the form  $L = L_1 + L_2 - V$ , where  $L_1$  and  $L_2$  are the Lagrangians of free subsystems and  $V$  accounts for the interaction part.

One of the key aspects of Soklakov's work is that complexity is physically interpreted as energy. In this paper we extend his ideas in order to ground the concept of entropy by means of the notion of complexity.

## 2 Mathematical background

This section is a brief review of some parts of [5].

Let  $X$  be the set of all finite binary strings  $\{\Lambda, 0, 1, 00, 01, 10, 111, 000, 001, \dots\}$ , where  $\Lambda$  is the string of length zero. Let  $Y$  be a subset of  $X$  such that no string in  $Y$  is a prefix of another. From now on we will consider only prefix computers, i.e., partial recursive functions  $C : Y \times X \rightarrow X$ . This is a very weak restriction from the theoretical point of view.  $C(p, d) = \alpha$  means that  $\alpha$  is the output of the computation of the data string  $d$  with the program string  $p$  by

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means of the computer  $C$ . The complexity of  $\alpha$ , given  $d$ , and relative to the computer  $C$ , is given by:

$$K_C(\alpha|d) = \min\{|p| \text{ such that } C(p, d) = \alpha\}, \quad (1)$$

where  $|p|$  denotes the length of the program  $p$  in bits. It is well known that there is an optimal computer  $U$  for which  $K_U(\alpha|d) \leq K_C(\alpha|d) + \kappa$ , where  $\kappa$  is a constant that depends on  $C$  and  $U$ , only. Any prefix computer can be simulated by  $U$ , and  $U$  is called a universal prefix computer.

In order to adopt a simpler notation, we make  $K(\alpha|d) = K_C(\alpha|d)$ .

One possible intuitive meaning for  $K(\alpha|d)$  is that it corresponds to the big picture of a very detailed object  $\alpha$  with a previous past given by  $d$ . It is worth to remark that the term “past” does not entail any corresponding notion of time. The point is that there is some kind of causation between  $d$  and  $\alpha$ . In [5] Soklakov describes this causation by means of time. But in this paper we use another parameter (with another interpretation) to relate  $d$  and  $\alpha$ .

Let  $^Jg : X \rightarrow X_1 \times \dots \times X_J (X_J = X)$  be a function. The complexity of  $^Jg$  at  $x_0 \in X$  is defined as

$$K_{x_0}[^Jg] = \frac{1}{J} \sum_{k=0}^{J-1} K(x_{k+1}|x_k), \quad (2)$$

where  $\{x_1, x_2, \dots, x_J\} = ^Jg(x_0)$  and  $K(x_{k+1}|x_k)$  is the complexity of  $x_{k+1}$  given data  $x_k$ , with respect to a universal prefix computer.

This last equation will be very important in our derivation of the second law of thermodynamics.

### 3 Entropy

In this section we intend to use complexity theory in thermodynamics. The first important problem is the physical interpretation of the involved mathematical concepts. Since we are interested on thermal systems, we interpret the strings of  $X$  as possible microstates of a given thermal system. So, let

$$K_{x_0}^{t_i}[^Tg] = \frac{1}{T} \sum_{k=0}^{T-1} K^{t_i}(x_{k+1}|x_k) \quad (3)$$

denote the complexity of a thermal system at absolute zero (the temperature is ideally zero Kelvin), with respect to the instant of time  $t_i$  ( $i$  stands for initial).  $K^{t_i}(x_{k+1}|x_k)$  is the complexity of the microstate  $x_{k+1}$  given the microstate  $x_k$ , at the initial instant  $t_i$ . It is important to remark that function  $^Tg$  corresponds to a dynamical evolution which starts at zero Kelvin. This means that we are adopting temperature as a parameter for describing such a dynamics. In [5] Soklakov used time as the parameter for describing the dynamics of a mechanical

system. The fact that the summation starts at zero ( $k = 0$ ) just reflects the absolute nature of temperature. So,  $T$  denotes an absolute value of temperature, although we are not talking about the Kelvin measurement scale. This absolute scale of temperature is zero when the temperature is zero Kelvin. But the main difference is that  $T$  is measured in discrete quantities, since we are talking about a summation.

Now let

$$K_{x_0}^{t_f}[Tg] = \frac{1}{T} \sum_{k=0}^{T-1} K^{t_f}(x_{k+1}|x_k) \quad (4)$$

denote the complexity of a thermal system at absolute zero, with respect to the instant of time  $t_f$  ( $f$  stands for final). Again we insist that the term “complexity of a thermal system at absolute zero” does not mean that the thermal system has a zero Kelvin temperature at instant  $t_f$ . It just means that the dynamics of the thermal system started at the zero point.

According to equations (3) and (4) we have:

$$K_{x_0}^{t_f}[Tg] - K_{x_0}^{t_i}[Tg] = \frac{1}{T} \sum_{k=0}^{T-1} (K^{t_f}(x_{k+1}|x_k) - K^{t_i}(x_{k+1}|x_k)). \quad (5)$$

The physical interpretation of this equation seems to be quite natural. If the microstates  $x_k$  ( $k = 0, 1, 2, \dots, T-1$ ) give us the detailed description of the thermal system, then  $K_{x_0}^{t_f}[Tg]$  and  $K_{x_0}^{t_i}[Tg]$  give the big picture of the same system at different instants of time. This big picture is the macrostate of the thermal system, i.e., the resulting energy associated to the system. So, we interpret the left side as entropy, and the difference between summations on the right side corresponds to the heat absorbed by the thermal system during the time interval  $(t_i, t_f)$ . Instead of a statistical or probabilistic approach to thermodynamics, this suggests a computational approach with analogous results.

## 4 Remarks

We make here some final remarks:

1. In [5] the author uses a very important physical principle which he calls the *simplicity principle* (SP). According to SP “among all dynamical laws that are consistent with all the other axioms [of the theory], the laws with the smallest descriptive complexity predominate the system’s behavior”. Actually this is a version of Occam’s razor principle, according to which simple theories are more economical and are usually better suited for making predictions. Soklakov uses SP in order to justify why it is necessary to minimize a functional of action which is an integral of the Lagrangian of the mechanical system that he studies. In our paper we make no use of

SP. This is so because we are concerned with a difference of complexities and not with any complexity itself. Thus, any process of minimization seems to be unnecessary here.

2. Many authors make a relationship between entropy and the direction of time. In equation (5) this relationship is made explicitly.
3. Equation (5) is an explicit formula that relates microstate ( $x_k$ ) and macrostate (temperature and heat). Nevertheless, like temperature, heat (energy) is discrete, although equation (5) is valid for any unit of measurement for energy if the unit measurement for temperature remains absolute, i.e., it is zero at the zero absolute temperature.
4. There is no need of the use of differential and integral calculus in our approach. In [5] the author understands the necessity for a formulation of the Galilean relativity principle in a discrete form, since the main basis of complexity theory is discrete mathematics. According to him, this problem will be considered in future research.
5. Our main result is that entropy may be understood as the complexity of a thermal system.

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